The Enemy of My Enemy Is My Friend: A New Condition for Network Games *

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Abstract

Group formation tends to involve peer effects. I provide a new sufficient condition for the non-emptiness of the core of network formation games that involve pairwise complementarities between peers. My condition allows for a novel class of preferences that are relevant to political alliances and publicly open friendship networks.

Keywords: Network Games, Coalition Formation, Cooperative Games, Core

JEL Codes: C62, C68, C71, C78, D44, D47, D50

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The king who is situated anywhere immediately on the circumference of the conqueror's territory is termed the enemy. The king who is likewise situated close to the enemy, but separated from the conqueror only by the enemy, is termed the friend [of the conqueror].—Kautilya, Arthasastra

1 Introduction

It is well known that in the presence of general complementarities/substitutabilities, coalition formation games need not have a non-empty core,[1] an indicator for the efficiency and stability of formed groups. Stability can be important since it keeps markets robust and supports their long-term sustainability (Roth (2002)). Focusing on network games that involve pairwise complementarities between peers, this paper provides a novel sufficient condition that allows for a new class of preferences relevant to political alliances and friendship networks.[2]

To model coalition formation with complementarities between peers, I restrict attention to pairwise complementarities. To explain this restriction, consider three agents, A, B, and C, who are thinking of forming a coalition. I assume that the total benefits for A from forming a group of size three with B and C can be decomposed into the gains from the individual relationships with B and C and the gain from the indirect synergy (pairwise complementarity) between B and C, in an additively separable manner. Such preferences

[1]See Shapley and Scarf (1974). A simple example in a cooperative game is as follows. Suppose a surplus of 2.9 will be generated when any pair of the three agents form a coalition, and a surplus of 3.0 will be generated when all three agents form a coalition. Notice that the grand coalition is welfare-maximizing but two of the three agents always form a blocking coalition.

[2]For the related literature on two-sided markets, exchange economies, and optimal assignment problems *with complementarities*, see, e.g., Bikhchandani and Ostroy (2002); Bikhchandani et al. (2002); Ausubel et al. (2006); Sun and Yang (2006); Sun and Yang (2009); Ostrovsky (2008); Echenique and Oviedo (2006); Pycia (2012); Hatfield et al. (2013); Kojima et al. (2013); Azevedo et al. (2013); Sun and Yang (2014); Azevedo and Hatfield (2015); and Che et al. (2019).

with additively separable pairwise complementarities are called *binary quadratic program* (BQP) preferences.[3]

For instance, suppose A is the U.S., B is South Korea, and C is Japan. Partly for purposes of its national defense against countries such as North Korea, the U.S. maintains bilateral defensive alliances with South Korea and Japan. For the interoperability of militaristic cooperation among the three countries, the South Korea-Japan relation is important to the U.S. Indeed, when the tension between South Korea and Japan rose and South Korea almost withdrew from the General Security of Military Information Agreement with Japan in 2019, the U.S. made significant diplomatic efforts to reconcile the two sides since the withdrawal would have a negative effect on U.S. security interests.[4] [5] This exemplifies how the indirect relation between B and C matters for A if A has a relation with both B and C. The implicit assumption here is that a coalition is *transparent* to every member in the coalition. I exclude the situation in which A can secretly build individual relationships with B and C and avoid the indirect synergy that B and C together bring to the group.

In this paper, I provide a new sufficient condition for the non-emptiness of the core of network games. The definition of the core in this paper is the same as that in Jackson (2005) with stability (no blocking coalition) and efficiency based on players rather than links. As stated in Jackson (2005), the core of a network game provides "a natural look at how the allocation of value of a network can be taken together with the formation of a network,

[3]Ausubel et al. (1997), for example, uses a form of BQP preferences in spectrum auction settings with pairwise complementarities, while Bertsimas et al. (1999), Candogan et al. (2015), and Candogan et al. (2018) consider combinatorial auction problems.

[4]See, e.g., `https://www.japantimes.co.jp/news/2019/08/23/national/politics-diplomacy/japan-south-korea-gsomia-intelligence-pact/`.

[5]Another example is that only when the South-North Korean relations improved in 2000 did England enter into a formal relation with North Korea.

especially regarding accounting for coalitional incentives" (p. 145) and "a protocol-free way analyzing the simultaneous allocation of value and formation of a network" (p. 146). To avoid confusion, peer effects in this paper are different from externalities. While this paper allows the allocation rule to be influenced by alternative network structures, a form of externalities as in Jackson (2005), the value of a component of the network itself is not allowed to be influenced by the network structure outside of the component, unlike Navarro (2007).[6]

With BQP preferences, there are two main conditions for the existence of a non-empty core. One is that if two agents work out well (badly), then any other agent will enjoy (suffer) the indirect non-negative (non-positive), possibly heterogeneous, synergy effects when forming a relation with these two agents. I call this condition the *sign-consistency* condition. For example, if the relation between Japan and South Korea is negative, I assume that it brings non-positive synergy effects to all other countries including the U.S. when these countries actually form a relation with both Japan and South Korea. While it is entirely possible that a country rather welcomes the tension between the two countries, I assume that if this country actually forms a relation with both of the two countries, then this country inevitably incurs the negative synergy.

The second condition is that a valuation graph of agents that specifies surplus from a link between two agents features the principle that the enemy of my enemy is my friend (and the friend of my friend is my friend). Mathematically, this principle is translated into the condition whereby the graph can be partitioned into a pair of subgraphs in which each of the subgraphs consists of positive edges, but the two subgraphs are connected by negative edges. I call this condition the *sign-balance* condition. For instance, during the

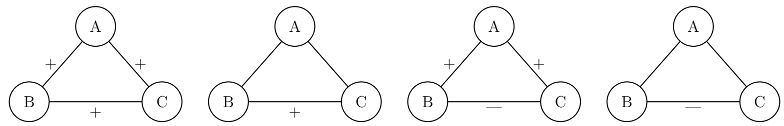
[6] Also, see Navarro (2010) for externalities on component-wise value functions.

cold war, the Soviet Union and North Korea were part of the East, while South Korea and the United States were part of the West. These countries were positively connected (allies) within each group and negatively connected (enemies) across the groups. Indeed, it has been established empirically that in addition to the state in which everyone is friends with each other, this relational state is most commonly observed in publicly open social networks such as individual human relations in massive online game experiments (Szell et al. (2010)); international relations (Maoz et al. (2007)); inter-gang violence (Nakamura et al. (2019)); trust/distrust networks among the users of a product review website (Facchetti et al. (2011)); friend/foes networks of a technological news site (Facchetti et al. (2011)); and elections of Wikipedia administrators (Facchetti et al. (2011)).

The existence result is obtained by demonstrating that the composite of the sign-consistency and sign-balance conditions with the BQP preferences implies the balancedness of a coalition formation game. The restriction to the BQP preferences and the sign-consistency condition allow me to explicitly express parameters on the positive or negative synergies between B and C from the perspective of A, and consequently allow me to identify how to group agents in order to prevent a blocking coalition. Then, I build a bridge between the sign-balance condition and the balancedness of a coalitional game by formulating the problem into a linear programming problem and by exploiting the properties of an extreme point solution.

Although it tends to be difficult to provide intuitive mathematical reasoning for approaches that use linear programming proofs, the intuition behind my main result is somewhat intuitive. Consider the four possible triads among agents A, B, and C, as depicted in Figure 1. Plus means a positive synergy, and minus a negative synergy. The case with all positives (sign-balanced) or all negatives (sign-unbalanced) is easy to solve; everyone forms

Figure 1: Examples of a cycle in balanced and unbalanced graphs



The two graphs on the left are balanced, and the other ones on the right are unbalanced

a relation with each other in the former, and no one forms a relation with each other in the latter. The second one from the left is the sign-balanced case with a common enemy. It is easy to imagine the stability of a formal relation between B and C; B and C will not form a relation with A and do not incur any damage from their negative synergies. Now, consider the third triad, with two positives and one negative, which is sign-unbalanced. In this case, B and C hope to form a relation with A, while A does not necessarily want to form a relation with both B and C. Thus, if only A and B form a relation, C wants to block this coalition and form a team with only A; if only A and C form a relation, then B has an incentive to block this coalition.[7]

Note that this paper does not intend to provide any *normative* arguments for such relations. For example, the West cooperated with Hitler, Mussolini, and Franco when its enemy of the 1930s was Stalin (Saperstein (2004)). Therefore, such a condition for stability does not justify any normative arguments for peace. Meanwhile, Maoz et al. (2007) empirically find that while there are many exceptions, enemies of enemies are three times more likely to become allies than by random chance.

The contributions of this paper are two-fold. The first is a technical contribution: I

[7] Meanwhile, the third case can be stable depending on the relative magnitude of the benefits from individual relationships to the damages from peer effects. If the benefits from individual relationships are larger than the damages from peer effects, then the coalition of interests may be stable. However, it is hard to precisely pin down such conditions. Consider a case in which an agent who already has 99 team members is trying to decide if she wants to bring in another member, with whom all of the other 99 members do not get along. Even if each of these negative peer effects is smaller than the positive direct value of forming a relationship with the 100th agent, the sum of all of the negative values can dominate the positive direct value.

demonstrate a connection between the sign-balance condition and a balanced game, two different concepts. The second contribution is that by exploiting BQP preferences, this paper studies network games that involve pairwise complementarities between peers and that are relevant to settings such as political group formation and friendship networks, and yields an existence result for the condition that has empirical support.

1.1 Related Literature

The sufficient condition for a non-empty core that is the closest to mine is the gross substitutes and complements (GSC) condition suggested by Sun and Yang (2006).[8] The GSC condition allows for preference structures more general than the preferences in this paper and is satisfied if goods can be divided into two groups, and within groups, goods are gross substitutes, and across groups, goods are gross complements[9]. However, in general, the condition would imply that a friend of my friend must be my enemy, which seems less plausible than mine in settings that involve humans (including countries and institutions).

Another set of papers close to mine are those by Candogan et al. (2015); Candogan et al. (2018); Nguyen and Vohra (2018); and Baldwin and Klemperer (2019). Nguyen and Vohra (2018) study two-sided matching markets with couples in the presence of capacity constraints, while allowing for general preferences in the context of nontransferable utility. In the presence of complementarities coupled with capacity constraints, their setting inevitably encounters potential emptiness of the core set, which they overcome by (possibly) minimally perturbing the capacity constraints. While their findings and algorithm are extremely powerful, direct

[8]The same-side substitutability and cross-side complementarity from Ostrovsky (2008) is similar.

[9]Note that while the GSC condition may sound *qualitatively* opposite to my condition, precisely speaking, it is not mathematically opposite in the class of preferences with additively separable pairwise complementarities.

application of their results to one-sided coalition formation is not immediate because of the lack of capacity constraints in my network game problems.

Baldwin and Klemperer (2019) provide a novel and powerful characterization of classes of valuations that result in Walrasian equilibria. In the sphere of their demand types, they provide necessary and sufficient conditions for such an equilibrium to exist. One may think that an appropriate basis change might allow their results to be applied to my one-sided coalition formation setting. However, this does not need to be the case. As demonstrated by an example in the Appendix that is same as Example 3.2 in the Supplemental Appendix from Candogan et al. (2015), the results of Baldwin and Klemperer (2019) do not allow for establishing the existence of a Walrasian equilibrium for sign-consistent tree (graph) valuations. Note that the sign-consistent tree valuation class in Candogan et al. (2015) is a subset of my sign-consistent sign-balanced valuation class. Thus, there may not be such a basis change for the results of Baldwin and Klemperer (2019) to be applicable to my model. Note that the results of Candogan et al. (2015) do not contradict the necessary and sufficient condition of Baldwin and Klemperer (2019) "since for sign-consistent tree valuations, it is implicitly the case that each item has a single copy, and while the equilibrium need not exist in the sense of [Definition 4.2 from Baldwin and Klemperer (2019)], it always exists when we restrict attention to this single-copy setting" (p. 34, Candogan et al. (2015) Online Appendix). Similarly, in my case, an agent cannot form multiple relationships with another agent, and implicitly, I assume no one agent is identical to another.

Candogan et al. (2015) provide necessary and sufficient conditions on agents' valuations to *guarantee* the existence of a Walrasian equilibrium in a case of one auctioneer and many bidders with multiple items[10]. They also exploit the BQP preferences and employ similar proof strategies for existence results. Furthermore, their conditions—the sign-consistency and tree structure on agents' valuation graph for a bundle of commodities—are similar to my conditions. I employ the same sign-consistency assumption, while I expand their tree valuation graph restriction to a sign-balance valuation graph that is a superset of theirs. For instance, the tree condition requires that if there are three goods A, B, and C, and non-zero complementarity/substitutability value attached to {A, B} pair and {A, C} pair, then there cannot be a non-zero complementarity/substitutability value attached to {B, C} pair to ensure the absence of a cycle in the entire graph. On the other hand, my sign-balance condition allows for a cycle (and a tree of course), while it restricts the structure of each cycle.

Aside from the difference in the domain of valuation, the major difference is that Candogan et al. (2015) focus on welfare-maximizing Walrasian equilibria of one-seller-many-buyer economies, while I focus on the core of one-sided coaltiion formation games. In the presence of complementarities, as implied in Shapley and Scarf (1974), a Walrasian equilibrium may not have the core property. In coalition formation settings, the literature has paid greater attention to the possibility of a blocking coalition formation and thus the core/stability property. Therefore, in settings that concern team and political group formation, I believe the core is the most suitable property to analyze.

Candogan et al. (2018) provide powerful results whereby within BQP preferences and a

[10]To avoid confusion, although Candogan et al. (2015) state that they "establish that the sign-consistency and tree graph assumptions are necessary and sufficient for our existence results," by providing examples in which violating one of the assumptions can lead to the non-existence of a Walrasian equilibrium, these assumptions are, strictly speaking, not technically necessary. This is because there can be many instances without the assumptions that have a Walrasian economy. What they really mean is that violating one of their conditions *can* lead to a lack of a Walrasian equilibrium.

more generalized version of these preferences, there always exists a certain pricing scheme to clear the market of a one-seller-many-buyer economy, as long as the pricing scheme is as complex as the preference structures. Note that this paper's one-sided coalition formation settings are outside the scope of their one-seller-many-buyer settings. Thus, their results are not directly applicable to my settings. Furthermore, my main result is applicable to both transferable utility and non-transferable utility games, and therefore, applicable to settings in which there is no market price (e.g., friendship, international relations, etc.).

2 Environment

Let $N = \{1, \ldots, n\}$ be a finite set of agents, considered fixed in what follows. A network is an undirected graph that is a list of unordered pairs of players $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked under the network g. When it is unambiguous, I write ij to represent $\{i, j\}$. Let g^N be the complete network (the set of all subsets of N of size two) on N. Denote by $G = \left\{g \mid g \subset g^N\right\}$ the set of all possible networks on N. For example, if $N = \{1, 2, 3\}$ and $g = \{12, 23\}$, then there is a link between players 1 and 2, a link between players 2 and 3, but no link between players 1 and 3. Furthermore, let $N(g)$ be the set of players who have at least one link in g.

Fix $S \subseteq N$. Let $g|_S = \{ij \in g : i \in S \text{ and } j \in S\}$ be the restriction of a given network g to subset S. A *path* in a network $g \in G$ between players i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$. A *component* of a network g is a non-empty subnetwork $g' \subset g$ such that (1) if $i \in N\left(g'\right)$ and $j \in N\left(g'\right)$ where $j \neq i$, then there exists a path in g' between i and j, and (2) if $i \in N\left(g'\right)$ and $ij \in g$, then $ij \in g'$. Denote by $C(g)$ the set of components of g.

A *value function* is a function $v : G \to \mathbb{R}$ and determines the total value that is generated by a given network structure. The set of all possible value functions is denoted by V. As noted in Jackson (2005), the calculation of value may include both costs and benefits and is a richer object than a characteristic function of a cooperative game since it allows the value that accrues to depend on both the coalition of players involved and the network structure. A value function v is *component additive* if $v(g) = \sum_{g' \in C(g)} v\left(g'\right)$ for any $g \in G$. This paper restricts attention to the "interesting sub-class of value functions" in which the value to a given component of a network is independent of the structure of other components (Jackson (2005), p. 132). This precludes externalities across (but not within) components of a network—i.e., I focus on component additive value functions.

A network $g \in G$ is *efficient* relative to a value function v if $v(g) \geqslant v\left(g'\right)$ for all $g' \in G$. Given a value function v, its *monotonic cover* $\hat{v}$ is defined by $\hat{v}(g) = \max_{g' \subset g} v\left(g'\right)$. A *network game* is a pair (N, v).

An *allocation rule* is a function $Y : G \times V \to \mathbb{R}^n$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g. An allocation rule determines how the value generated by a network is allocated among the players, either through their decisions or perhaps even by some outside intervention. A network allocation pair $g \subset g^N$ and (imputation) $y \in \mathbb{R}^n$ is in the *core* of the network game (N, v) if $\sum_i y_i \leqslant v(g)$ and $\sum_{i \in S} y_i \geqslant \hat{v}\left(g^S\right)$ for all $S \subset N$. An allocation rule Y is *core consistent*, if for any v such that the core is non-empty, there exists at least one g such that $(g, Y(g, v))$ is in the core.

Additionally, to describe sufficient conditions and obtain the existence result, I shall impose more structures—i.e., BQP preferences—on value functions. The exogenously given values of match surplus among every pair of agents from the perspective of agent i are represented by a (complete) undirected *valuation graph* W^i, where $w^i_{ij} \in \mathbb{R}$ for $i \neq j$ captures the value of a link with agent j from the perspective of i, while $w^i_{jk} \in \mathbb{R}$ for $i \neq j \neq k$ represents a pairwise complementarity between potential peers j and k for i. I assume that $w^i_{ii} = w^i_{jj} = 0$ for any i and j. A valuation graph for agent i is comprised of (N, E^i) where E^i contains a value edge (i, j) whenever $w^i_{ij} \neq 0$ and a value edge (j, k) whenever $w^i_{jk} \neq 0$. Notice that the value of each pair of agents can be different among different agents, to account for heterogeneity in valuation and the costs of maintaining such relations. Note that I assume BQP preference structures on value function such that $\sum_i \left(\sum_{j:\{i,j\}\in g|_S} w^i_{ij} + \sum_{j,k:\{i,j\},\{i,k\}\in g|_S} w^i_{jk} \right) = v(g|_S)$ for any $S \subseteq N$. Let W be the collection of all agents' valuation graphs.

Let $M(W^i) = \{i | \exists j \text{ s.t. } w^i_{ij} \neq 0\}$. A *value path* in W^i connecting i_1 and i_n in value is a set of distinct nodes $\{i_1, i_2, \dots, i_n\} \subset M(W^i)$ such that $w^i_{i_1 i_2}, w^i_{i_2 i_3}, \dots, w^i_{i_{n-1} i_n} \neq 0$. A *cycle* is a value path in which no node except the first, which is also the last, appears more than once.

3 Existence

I first introduce the so-called sign-consistency assumption introduced by Candogan et al. (2015). The idea is that if two agents work out well (badly), then any other agent will enjoy (suffer) the indirect non-negative (non-positive) synergy effects when forming a relation with these two agents.

Assumption 3.1. *(Sign Consistency). For some $i, j \in N$, if $w^i_{ij} > 0$, then $w^k_{ij} \geq 0$ for all $k \in N$, and similarly, if $w^i_{ij} < 0$, then $w^l_{ij} \leq 0$ for all $l \in N$.*

For example, if the relation between Japan and South Korea is negative, I assume that it brings non-positive synergy effects to all other countries when these countries actually form a relation with both Japan and South Korea. While it is entirely possible that a country rather welcomes the tension between the two countries, I assume that if this country actually forms a relation with both of the two countries, then this country inevitably incurs the negative synergy.

Next, I introduce the so-called sign-balance assumption.[11] Colloquially, the condition requires that the enemy of my enemy is my friend. The important property of a sign-balance graph is so-called *clusterability* (Cartwright and Harary (1956)); one can regroup the nodes of the graph into two subgroups within which $w_{ij}^k > 0$ or $w_{ij}^k = 0$ and across which $w_{ij}^k < 0$. This seems qualitatively the opposite of the GSC condition, although the two conditions are mathematically not opposite due to the numerical restrictions of the GSC condition under BQP preferences.[12] Figure 1 shows examples of a sign-balance graph. The two graphs on the left are sign balanced, while the two on the right are not.

Assumption 3.2. *(Sign Balance). For every i, any cycle in W^i contains an even number of negative edges.*

My proof for the main result exploits the primal-dual relation between welfare-maximizing solutions and the core. In particular, I first show that the following quadratic program (QP1) has an integer-value solution:

[11]This is sometimes called the "structural balance condition."

[12]See Murota (2003); Murota and Shioura (2004); Iwamasa (2018); and Koizumi (2019).

$$\text{maximize} \sum_{k \in N} \left(\sum_{i \neq k} w_{ik}^k x_i^k + \sum_{i \neq j \neq k} w_{ij}^k x_i^k x_j^k \right)$$

$$\text{subject to } x_i^k = x_k^i \quad \forall i, k \in N,$$

$$0 \leq x_i^k \leq 1 \quad \forall i, k \in N,$$

where $x_i^k = 1$ if agent i is matched to agent k, $0 < x_i^k < 1$ if *a fraction* of agent i is matched to agent k, and $x_i^k = 0$ if agent i does not form a relation with agent k. For convenience, define $x_k^k = 0$ for all $k \in N$. The constraint,$0 \leq x_i^k \leq 1$ $\forall i, k \in N$ implies $\sum_{i \in N} x_i^k \leq N$ $\forall k \in N$, which ensures that agent k forms relations with no more than N agents including herself.

I prove the existence of an integral solution by extending the version of the proof for the tree-valuation graph from Candogan et al. (2015) written in one of Vohra's blog posts (2014).[13] His proof uses induction, and in particular shows that an extreme point in the polyhedron of the welfare-maximizing problem formulated in the linear programming manner is integral for every natural number of the cardinality of the maximal connected components of the valuation graph after deleting negative edges. Exploiting the tree structure, he divides the graph into one connected component and the complement of the component, which allows him to formulate the original problem as the convex combination of the two parts of the graph, both of which have integral solutions.

The proof for the one-sided coalition formation setting turns out to be much simpler, since the setting does not involve typical constraints of one good to one agent from buyer-seller or

[13] https://theoryclass.wordpress.com/2014/02/10/combinatorial-auctions-and-binary-quadratic-valuations-postscript/

optimal assignment problems. Using the sign-consistency assumption, we can categorize each edge from each agent's value graph into either non-negative or strictly negative. Combined with the sign-balance assumptions, the absence of such constraints divides the induction problem into two simple cases, ignoring the degenerate case in which all of the edges are negative. The first case is that the sign of all the edges is all non-negative. In this case, the solution is easy since matching everyone to each other (i.e., the complete network) is the solution, and thus there exists a solution to (QP1) that is integral.

The second case involves the clusterability of the sign-balance graph. By this property, we can partition any value graph into two subsets, within which edges are all non-negative and across which the edges are all negative. Then, one can simply extend a portion of Vohra's proof to this case with the following two modifications. First, for the n cardinality case, unlike the tree-valuation graph, my graph may not have a component of the maximal connected components that has a node with exactly one negative edge to a node in one of the other maximal connected components; rather, a node in such a component can be incident to multiple negative edges. Furthermore, within this component, there may be multiple nodes that are connected to other components with negative edges. The proof for the following lemma can be found in the Appendix.

Lemma 1. *Let Assumption 3.1 and 3.2 hold. Then, (QP1) has an integral solution.*

Note that as implied in Shapley and Scarf (1974) and discussed in Demange (2004), this is not enough to prove that the welfare-maximizing allocation actually has the core property. With Lemma 1, my main theorem can be obtained by linearizing (QP1) and applying the primal-dual approach of a linear programming framework. My proof uses an equivalent formulation of the original primal problem whose dual does not immediately correspond to the core. I first render this step as a lemma. In particular, I claim that the following linear program named (P1) is an equivalent formulation to the linearly relaxed formulation of the original problem, (LP1), when solutions are restricted to extreme points:

$$H(N) = \max \sum_{S^2 \subseteq [N]^2} v\left(S^2\right) x(S^2)$$

$$\text{subject to} \sum_{S^2 \ni (i,j)} x(S^2) \leq 1 \;\; \forall (i,j) \in [N]^2$$

$$x(S^2) \geq 0 \;\; \forall S^2 \subseteq [N]^2,$$

where S^2 is a subset of the size-two order-free power set of N, denoted by $[N]^2$—i.e., $\{1,2\}$ and $\{2,1\}$ are considered to be equivalent sets and do not simultaneously lie in the power set—, $v(S^2) = \sum_{i \neq k \in S^2} w_{ik}^k + \sum_{i \neq j \neq k \in S^2} w_{ij}^k$ with some abuse of notation and $x(S^2)$ indicates an integral or fraction of S that form a relation. For example, if $S = \{\{1,2\},\{2,3\}\}$, and if $x(S^2) = 1$, then agents 1 and 2 form a relation and agents 2 and 3 form a relation, while agents 1 and 3 do not. Notice that when a solution is integral, then $H(N) = \hat{v}(g)$ for g induced by an assignment vector x with entries for each non-empty size two subset S^2, $x(S^2)$.

The equivalence is immediate by showing the one-to-one mapping between (P1) and (LP1) with extreme point solutions, and thus the proof is omitted from this paper.

Lemma 2. *Let Assumption 3.1 and 3.2 hold. Then, when solutions are restricted to extreme points, (P1) is an equivalent formulation to the linearly-relaxed formulation of (QP1).*

Note that the dual of (P1) does not immediately correspond to the core, either. To find the primal program of the dual that does correspond to the core, I apply balancing weights from Bondareva (1963) and Shapley (1967) to bridge (P1) to such a primal problem. In this sense, as far as I know, this is the first study to find a connection between sign-balanced graphs and balanced games, which are different concepts. My technique provides a way for future research to find a point in the core when researchers study one-sided formation problems with complementarities that are beyond the existing class of complementarities.

Theorem 1. *Let Assumption 3.1 and 3.2 hold. Then, the core is non-empty.*

Proof. Using the objective value of (P1), we can construct another linear program (P2) with its dual (DP2) as below:

P2

$$\max \sum_{T \subseteq N} H(T)\, y(T)$$
$$\text{subject to} \sum_{T \ni i} y(T) = 1 \quad \forall i \in N$$
$$y(T) \geq 0, \quad \forall T \subseteq N \setminus \varnothing$$

DP2

$$Z(N) = \min \sum_{i \in N} \pi^k$$
$$\text{subject to} \sum_{i \in T} \pi_i \geq H(T)$$
$$\forall T \subseteq N \setminus \varnothing$$
$$\pi_i \geq 0 \;\; \forall i \in N$$

Notice that a solution with $y(T) = 1$ for $T = N \setminus \varnothing$ and with $y(T') = 0$ for all $T' \subset T$ is a solution to (P2). Otherwise, there exists no extreme point solution in (P1). Similarly, for a game with any restricted subset $S \subset N \setminus \varnothing$, $y(S) = 1$ and $y(S') = 0$ for all $S' \subset S$ is a solution to (P2) as well. Thus, the objective value of (P1) equals that of (P2). Note that this logic is valid only because we started with (P1) that is *not* an integer program.

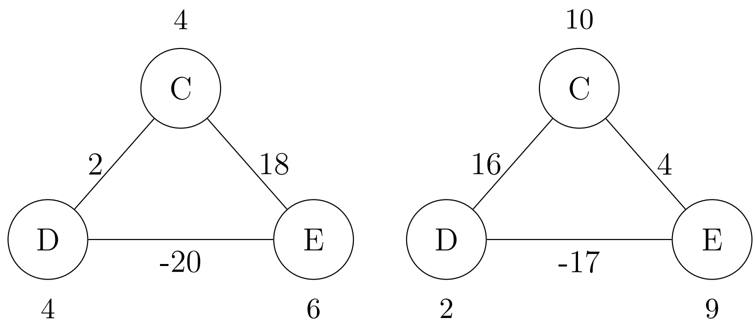
Meanwhile, notice that (DP2) corresponds to the core. Take an optimal solution to (DP1), $(\pi^\star)$, and consider a subset of agents R. Denote by $(\pi^\star(R))$ an optimal solution to the dual when restricted to subset R. Now, we can compute the objective value of the dual for a subset of agents R, $Z(R) = \sum_{k\in R} \pi^{k^\star} \geq \sum_{T\subseteq R} H(T)y(T) = H(R) = \hat{v}(g|_S)$ by weak duality and the equality of the objective values between (P1) and (P2). Now, by strong duality (coming from the integrality of a solution to (P2)), $\sum_{k\in N} \pi^{k^\star} = \sum_{T\subseteq N} H(T)y(T) = H(N) = \hat{v}(g)$. This implies that by Lemma 1 and 2, with an extreme point solution to (P1) that is also a solution to (LP1) and (QP1), there is a system of imputations, $(\pi^\star)$, to assign an allocation to every agent that results in the core, given that this system of imputations comes from an extreme point solution. Thus, the pair of network g induced by the extreme point solution and the system of imputations lies in the core. ■

It is immediate from Table 1 of Jackson (2005) that the player-based networkolus is a core-consistent allocation. To define the player-based networkolus, let $e^p_S(y) = \sum_{i\in S} y_i - \hat{v}\left(g^S\right)$ be the excess allocated to coalition S relative to their monotonic value under v, and denote by $e(y)$ the vector with entries for each non-empty $S \subset N$. Let $\phi^{\mathrm{Nuc}}(\hat{v}) = y$ be the unique allocation such that $e^p(y)$ leximin dominates $e^p\left(y'\right)$ for all y' such that $\sum_i y'_i = \hat{v}\left(g^N\right)$. Then, the player-based networkolus is

$$Y^{\mathrm{PN}}(g, v) = \frac{v(g)}{\hat{v}\left(g^N\right)}\phi^{Nuc}(\hat{v}).$$

Corollary 1. *Let Assumption 3.1 and 3.2 hold. Then, there exists a network-allocation pair* $\left(g, Y^{\mathrm{PN}}\right)$ *that lies in the core.*

Figure 2: Example without a core

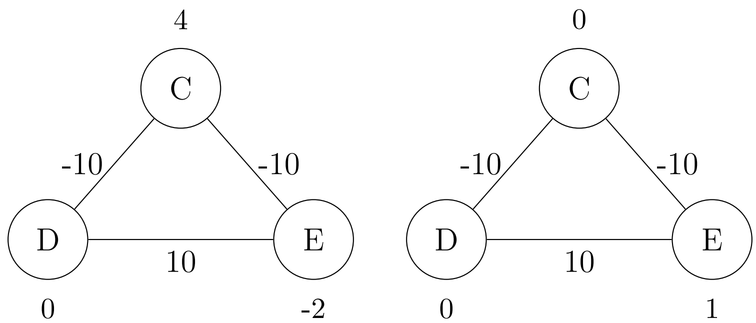


The left graph corresponds to agent A's valuation graph when restricted to agent C, D, and E, and the right graph corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$.

4 Example

In this section, I shall show how violation of the sign-balance condition can result in an empty core. Suppose we have two agents A and B with $w^A_{AB} = w^B_{AB} = -100$. Furthermore, suppose there are thee more agents, C, D, and E with which agent A or B is considering forming a relation, while agents D, E, and F are indifferent in forming relations with each other—i.e., $w^k_{\text{CD}} = w^k_{\text{CE}} = w^k_{\text{DE}} = 0$ for $k \in \{\text{C, D, E}\}$. Moreover, from these three agents' perspectives, there is no intrinsic benefit of forming a relation with A or B—i.e., $w^k_{\text{A}k} = w^k_{\text{B}k} = 0$ for $k \in \{\text{D, E, F}\}$. Finally, assume that w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$ and w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$ are depicted by Figure 2. The graph on the left corresponds to agents A's value graph when restricted to agent C, D, and E, and the graph on the right corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$. In this example, any of the three agents C, D, and E will not form a coalition with *both* A and B, due to the prohibitively high negative synergy between A and B.

Figure 3: Example with a core



The left graph corresponds to agent A's valuation graph when restricted to agent C, D, and E, and the right graph corresponds to that of B. The number above/below each node represents w_{ki}^k for $k \in \{A, B\}$ and $i \in \{C, D, E\}$, while the number above each edge represents w_{ij}^k for $k \in \{A, B\}$ and $(i, j) \in \{\{C, D\}, \{C, E\}, \{D, E\}\}$.

This example has an empty core. Just to illustrate how the infinite loop of blocking occurs, let us look at an arbitrary start of this loop. Suppose agent A forms a relation with C and E together, while agent B forms a relation with agent D. At a glance, this seems to be an efficient outcome and thus achieves no blocking coalition. And yet, notice that if agent A pays agent C less than 26, then agent C forms a blocking coalition with agent B and D. So, agent A has to pay agent C 26, and pays agent E no more than 2 since otherwise, agent A would obtain a negative payoff. But then, agent B will leave agent D and form a blocking coalition with agent E, paying her any amount in (2, 7) (since agent B can get at most 2 from matching with agent D).

Similar blocking processes will happen at any combination of coalition formation among these five agents, and thus this is an example with an empty core when the sign-balance condition is violated.

Next, I shall provide an example with a core. Consider the valuation graphs of A and B as illustrated in Figure 4 that satisfy the sign-consistency and sign-balance conditions. In this case, A forms a coalition with C, while B forms a coalition with D and E.

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Appendices

A Proof of Lemma 1

Proof. Suppose L is a valuation graph with node set N and suppose any $i, j \in N$ such that for any $k \in N$, $w_{ij}^k \neq 0$ introduces edge (i, j). By sign consistency, we can label such edges as positive or negative based on the sign of w_{ij}^k. Let $E^+ = \{(i, j) : w_{ij}^k \geq 0 \text{ for some } k \in N\}$ and $E^- = \{(i, j) : w_{ij}^k < 0 \text{ for some } k \in N\}$. Notice that any edge (i, j) lies in the set of negative edges even when only one agent regards it as negative and the remaining agents regard it as zero. Now, by the clusterability of the sign-balance condition, we can partition the nodes of L into two subsets, L_1 and L_2, called plus-sets, such that each subset only contains positive edges (including zero) and across the subsets, the edges are all strictly negative. If either L_1 or L_2 is the empty set, then the solution is easy. Every agent forms a relation with the rest, and we achieve the maximum, which implies that the solution is integeral. If both are the empty set, then a zero vector is the solution and therefore, the solution is integral.

Thus, suppose L_1 and L_2 are nonempty sets. To solve this case, I first introduce a new variable, z_{ij}^k, to linearize the quadratic terms in (QP1):

$$\begin{aligned} \text{maximize} & \sum_{k \in N} \left(\sum_{i \neq k} w_{ik}^k x_i^k + \sum_{i \neq j \neq k} w_{ij}^k z_{ij}^k \right) \\ \text{subject to } & x_i^k \leq 1 \;\; \forall i, k \in N \\ & x_i^k = x_k^i \;\; \forall i, k \in N \\ & z_{ij}^k \leq x_i^k, x_j^k \;\; \forall k \in N, (i,j) \in E_+^k \\ & z_{ij}^k \geq x_i^k + x_j^k - 1 \;\; \forall k \in N, (i,j) \in E_-^k \\ & x_i^k, z_{ij}^k \geq 0 \;\; \forall i, j, k \in N \end{aligned}$$

Note that we can formulate in this way, due to Bertsimas et al. (1999). We call this relaxed formulation (LP1). Let P_0 be the polyhedron of feasible solutions to (LP1). The goal is to show the extreme points of P_0 are integral, which implies there exists a feasible optimal solution that is integral. The way to do this is to use the fact that if the constraints matrix is totally unimodular, then the extreme points of P_0 are integral.

Let $(\bar{z}, \bar{x})$ be an optimal solution to (LP1). We can choose it to be an extreme point of the corresponding polyhedron P_0 of (LP1). Also, let P be the polyhedron restricted to the nodes of L_1 and let P' be that restricted to the vertices of L_2. Then, consider any node p of L_1 that is connected to a proper subset of the members of the other group, say $Q \ni q$.

We know the sign of the edge (p, q). By the logic from the case in which L_1 or L_2 is the empty set, both P and P' are integral polyhedrons. Now, let $X_1, ..., X_n$ be the set of extreme points of P for some natural number n while $Y_1, ..., Y_{n'}$ be that of P' for some natural number n'. Let $v(.)$ be the objective value of any extreme point X_r or Y_r.

Since a polyhedron is convex, we can express $(\bar{z}, \bar{x})$ restricted to $\boldsymbol{P}$ as $\sum_r \lambda_r X_r$ while $(\bar{z}, \bar{x})$ restricted to $\boldsymbol{P}'$ as $= \sum_r \zeta_r Y_r$. Let $\boldsymbol{E}_-$ as the set of negative edges restricted to those involving the vertices in $\boldsymbol{L}_1$. Then, we can rewrite (LP1) as:

$$
\begin{aligned}
\text{maximize} & \sum_r \lambda_r v(X_r) + \sum_r \zeta_r v(Y_r) - \sum_{k \in L_1} \sum_{(p,q) \in E_-} |w^k_{pq}| y^k_{pq} \\
\text{subject to} & \sum_r \lambda_r = 1 \\
& \sum_r \zeta^q_r = 1 \\
& y^k_{pq} \leq 1 \quad \forall k \in L_1, (p,q) \in E_- \\
& \lambda^p_r, \zeta^q_r, y^k_{pq} \geq 0 \ \forall r, k
\end{aligned}
$$

Notice that the constraint matrix of this linear program is again a network matrix, and thus totally unimodular. This is because each variable appears in at most one constraint with a coefficient of 1. Therefore, there exists an integral solution in this program. ■

B Example 3.2 of the Online Appendix of Candogan et al. (2015)

This section introduces Example 3.2 in the Online Appendix of Candogan et al. (2015) that studies one-seller-many-buyer settings whose basis is closer than mine to Baldwin and Klemperer (2019). First, I summarize the main results of Baldwin and Klemperer (2019), and then present an example that does not lie in their demand types but still constitutes a competitive equilibrium due to the smaller domain space for the number of each item.

In their paper, they define a concavity property of valuations and the notion of demand type that are used for the characterization of Walrasian equilibria. Suppose there is one seller, I buyers, and N goods. The concavity condition is satisfied by u if and only if for each bundle of goods S in the domain of valuations, there exists a price vector p such that the associated set of demand bundles $D_u(p)$ contains S. This concavity condition is satisfied by the monotonicity assumption that more goods are better, which is assumed in Candogan et al. (2015).

The demand type is defined by tropical hypersurfaces associated with demand sets. Tropical hypersurfaces are the set of prices, $T_v(p) = \left\{p \in \mathbb{R}^N || D_u(p)| > 1\right\}$; in other words, the set of prices at which multiple bundles are demanded. This hypersurface defines a geometric object that separates different regions of the price space in which only a single bundle is demanded. The primitive integer normals corresponding to the facets of this geometric object capture how demand varies from one region to another. This set of normals characterizes an agent's demand type, formally defined by Baldwin and Klemperer (2014) and equivalent to Definition 3.1 from Baldwin and Klemperer (2019), as follows:

Definition B.1. An agent has demand of type D if all of the primitive integer normals to the facets of the tropical hypersurface of its demand lie in the set D.

To make this more chewable, these normals basically capture how demand changes as prices change. For example, suppose there are three items i, j, k and suppose at price vector p, bundle $\{i\}$ is demanded. Furthermore, suppose as the price of i increases, bundle $\{j, k\}$ starts being demanded. In this case, if at the price vector at which bundles $\{i\}$ and $\{j, k\}$

are both demanded, no other bundle is demanded, then the demand type involves vector $[-1, 1, 1]$, where the entries of this vector correspond to index $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Now, we need to somehow aggregate individual valuation. The powerful existence results of Baldwin and Klemperer (2019) implicitly use a *strong* definition of existence of equilibrium for a class of valuations, in the sense that their competitive equilibrium with aggregate valuation requires an equilibrium to exist for *any* choice of valuations and *any* number of copies of items consistent with the aggregate valuation function[14]. Note that the domain of the aggregate valuation captures the total demand by all buyers; for instance, if all buyers demand all items at a certain price, then this set will allow for $\boldsymbol{I}$ copies of each item). Then, if an equilibrium does not exist for some set of valuations with demand type $\boldsymbol{D}$ and some supplies of each item, their equilibrium definition suggests that an equilibrium does not exist. Notice that if there are restrictions on the number of copies of each item, then an equilibrium may exist. This is indeed the case for sign-consistent tree valuations.

Now, let us look at their formal definition of unimodular demand type:

Definition B.2. A demand type $\boldsymbol{D}$ is unimodular if any linearly independent set of vectors in $\boldsymbol{D}$ is an integer basis for the subspace they span.

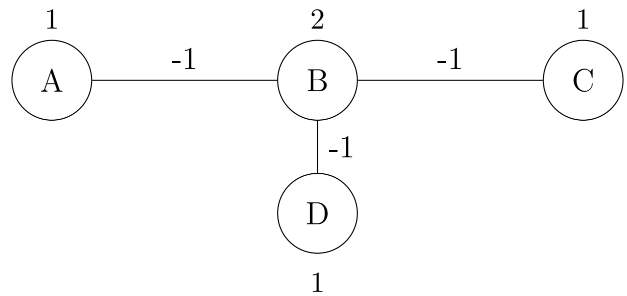
With this definition, I introduce their main results:

Theorem 2. *(Unimodularity Theorem): An equilibrium exists for every pair of concave valuations of demand type* $\boldsymbol{D}$*, for all relevant supply bundles, iff* $\boldsymbol{D}$ *is unimodular.*

Corollary 2. *With* $\boldsymbol{n}$ *goods, if the vectors of* $\boldsymbol{D}$ *span* $\mathbb{R}$*, then an equilibrium exists for every*

[14]Baldwin and Klemperer (2019) note that the unimodular theorem "states that competitive equilibrium always exists, whatever is the market supply, if and only if [...]" (p. 868).

Figure 4: Example 3.2 from the Online Appendix of Candogan et al. (2015)



Numbers above or below the nodes represent individual surplus terms w_l from obtaining good l and those above or next to the edges represent pairwise surplus terms w_{lk} from obtaining a pair of goods l and k.

finite set of concave valuations of demand type D*, for all relevant supplies, iff every subset of* n *vectors from* D *has determinant* 0 *or* ± 1.

Before introducing the example, I introduce a payoff function of buyers. Suppose the seller wants to maximize her revenue and buyer i has the following BQP preferences from obtaining a bundle of goods S:

$$u_i(S) = \sum_{j \in S} (w^i_j - p_j) + \sum_{j,k \in S : j \neq k \neq i} w^i_{jk}, \tag{1}$$

where the difference from the main text is a uniform price of good j across buyers.

Now, consider a situation in which there are four goods, A, B, C, and D. To simplify the setting, suppose all of the buyers have the same preferences over these four goods. Figure 3 demonstrates the buyers' preferences. Numbers above or below the nodes represent individual surplus terms w_l from obtaining good l and those above or below the edges represent pairwise surplus terms w_{lk} from obtaining a pair of goods l and k.

Next, to check the demand type associated with the valuation function, I label the entries of the demand type vectors by A, B, C, and D, respectively. First, assume that the price of item D is arbitrarily high, and prices of the remaining goods are $p(\mathrm{A}) = 0.1, p(\mathrm{B}) = 0.5, p(\mathrm{C}) = 0.1$. Then, it follows that the corresponding demand $\boldsymbol{D}(p) = \{\mathrm{A}, \mathrm{C}\}$. Now, change the price to $p'(\mathrm{A}) = 1, p'(\mathrm{B}) = 0.5, p'(\mathrm{C}) = 0.1$. It follows that the demand $\boldsymbol{D}(p') = \{\mathrm{B}\}$. This change in the demand set implies that vector $d_1 = [1, -1, 1, 0]$ lies in the demand type associated with the valuation. Using the symmetry between the nodes, we can deduce that vectors $d_2 = [1, -1, 0, 1]$ and $d_1 = [0, -1, 1, 1]$ also lie in the demand type. Lastly, consider another price vector $p'' = [2, 0, 2, 2]$ at which the only demanded bundle is $\{\mathrm{B}\}$. Notice that if we increase the price of B, the demand set switches to the empty set. Therefore, the demand type also contains $d_4 = [0, -1, 0, 0]$. Observe that the matrix with columns d_1, d_2, d_3, and d_4 has determinant 2. Hence, it follows from Corollary 1 that the demand type associated with the tree valuation in Figure 3 does not always have a Walrasian equilibrium *if any supply bundle is allowed*. Yet, restricting preferences to sign-consistent tree valuations and the supply of each item to be a single copy, Candogan et al. (2015) finds a Walrasian equilibrium. Note that the sign-consistent tree valuation class is a subset of my sign-consistent sign-balanced valuation class.