The Enemy of My Enemy Is My Friend: New Conditions for Network Games *

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Abstract

Group formation tends to involve peer effects. In the presence of such complementarities, however, coalitional games need not have a nonempty core. With a restricted preference structure, I provide new sufficient conditions for the nonemptiness of the core of network games that involve pairwise complementarities between peers. The conditions are twofold: (a) sign-consistency—all agents agree on the sign of the value of any link—and (b) sign-balance—the enemy of my enemy is my friend. My conditions provide a game-theoretic explanation for the longevity of the dichotomy of political alliances in the contemporary world.

Keywords: Network Games, Coalition Formation, Cooperative Games, Core

JEL Codes: C62, C68, C71, C78, D44, D47, D50

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The king who is situated anywhere immediately on the circumference of the conqueror's territory is termed the enemy. The king who is likewise situated close to the enemy, but separated from the conqueror only by the enemy, is termed the friend [of the conqueror].—Kautilya's, Arthasastra

1 Introduction

It is well known that in the presence of general complementarities/substitutabilities, coalitional games need not have a nonempty core.[1] Since it is challenging to find conditions for the nonemptiness of a core with generality, one approach is to restrict preference structures. This is the approach I take in this paper. Focusing on network games that involve pairwise complementarities[2] between peers that are relevant to political alliances, this paper provides novel sufficient conditions for such games to have a nonempty core.

With pairwise complementarities, I focus on an additively separable payoff structure. Within this structure, agents individually obtain additive payoffs from both direct bilateral relations and indirect pairwise synergistic effects between peers.[3] Such preferences with additively separable pairwise complementarities are sometimes called *binary quadratic program* (BQP) preferences.[4]

As a real-life example, partly for purposes of its national defense against countries such as North Korea, the U.S. maintains bilateral defensive alliances with South Korea and Japan.

[1]See Shapley (1955) and Shapley and Scarf (1974).

[2]By pairwise complementarities, this paper means a complementarity between two agents from the perspective of another agent to contrast with more general complementarities, such as a complementarity generated by three or more agents from the perspective of another agent which can be captured by, e.g., a use of hypergraph.

[3]This paper uses "form a relation" and "form a link" interchangeably.

[4]Ausubel et al. (1997); Bertsimas et al. (1999); Candogan et al. (2015); and Candogan et al. (2018) consider combinatorial auction problems in which a bidder receives payoffs from both individual goods and synergies between a pair of goods in an additively separable manner. Their preference structures are categorized as BQP preferences.

For the interoperability of militaristic cooperation among the three countries, the South Korea-Japan relation is important to the U.S. When tension between South Korea and Japan arose in 2019 and South Korea almost withdrew from the General Security of Military Information Agreement with Japan, the U.S. made significant diplomatic efforts to reconcile the two sides, since the withdrawal would have a negative effect on U.S. security interests.[5] This exemplifies how the indirect synergy between peers matters for forming multiple bilateral relations. The implicit assumption here is that each formed bilateral relation is *transparent* to every agent in the game. I exclude the situation in which one agent can secretly build individual relationships with two agents and avoid the indirect synergy between them.

The definition of the core in this paper is the same as that in Jackson (2005), with stability (no blocking coalition) and efficiency based on agents rather than links. To avoid confusion, peer effects in this paper are different from externalities. While this paper allows the allocation rule to be influenced by alternative network structures, a form of externalities as in Jackson (2005), the value of a component of the network itself is not allowed to be influenced by the network structure outside of the component, which is different from Navarro (2007).[6]

With the restricted preference structure, there are two main conditions for the existence of a nonempty core. To explain them, consider a valuation graph of agents that specifies a surplus value from a potential link between two agents—i.e., the so-called "intrinsic value" from Jackson and Wolinsky (1996). The first condition is that all agents agree on the sign of the underlying surplus value of a potential link between any pair of agents. I call this

[5] See, e.g., `https://www.japantimes.co.jp/news/2019/08/23/national/politics-diplomacy/japan-south-korea-gsomia-intelligence-pact/`.

[6] Also, see Navarro (2010) for externalities on component-wise value functions.

condition the *sign-consistency* condition.

In this paper, surplus values are actual and are not values expected by the agents, to avoid potential disagreements among the agents with respect to the magnitudes of the values.

While a conflict or tension between two countries could possibly be "beneficial" to another country, I assume that if this country actually forms a relation with both of the countries, then this country inevitably incurs the negative synergy. For example, during the Cold War, Egypt at least ostensibly benefited from foreign aid competition between the Soviet Union and the U.S.[7] After all, Egypt terminated its relation with the Soviet Union in the 1970s with Sadat's termination of the Soviet-Egyptian Treaty of Friendship and Cooperation.[8]

The second condition is that for each agent, the associated valuation graph from this agent's perspective features the condition whereby the graph can be partitioned into a pair of subgraphs such that each of the subgraphs consists of links whose values are positive (positive edges), but the two subgraphs are connected by links whose values are negative (negative edges). I call this condition the *sign-balance* condition. This condition has an economically meaningful interpretation. For size-three coalitions, it is translated into the principle that the enemy of my enemy is my friend (and the friend of my friend is my friend). For each larger-size coalition, agents can be assigned to two groups, within which agents are friends (or neutral), but across which agents are enemies.[9]

[7] Listing the amounts of foreign aid provided by the Soviet Union and U.S. to Egypt during the Cold War, Tomita (1995) points out that Egypt relied upon foreign aid as a driving force of its growth and exploited the international politics back then.

[8] See `https://www.nytimes.com/1976/03/15/archives/new-jersey-pages-sadat-acts-to-end-pact-with-soviet-cairo-signed-in.html`.

[9] It has been established empirically that in addition to the state in which everyone is friends with each other, the state in which the enemy of my enemy is my friend is most commonly observed in international relations (Maoz et al. (2007)); publicly open social networks such as individual human relations in massive online game experiments (Szell et al. (2010)); inter-gang violence (Nakamura et al. (2019)); trust/distrust networks among the users of a product review website (Facchetti et al. (2011)); friends/foes networks of a technological news site (Facchetti et al. (2011)); and elections of Wikipedia administrators (Facchetti et al. (2011)).

The existence result is obtained in the following way. The restriction of preferences to the additively separable payoff structure with pairwise complementarities and the sign-consistency condition allow me to explicitly express parameters on the positive or negative synergies between B and C from the perspective of A. Furthermore, with the sign-balance condition, one can categorize agents into a pair of groups as described above. Then, one can let all of the agents be linked to each other within groups and not linked to anyone across groups, if there are no large enough synergies for any agent that offset the negatives across groups. For cases with large enough synergies that incentivize agents to form a relation across groups, I use a linear programming approach to ensure no infinite loop of blocking coalitions that hinder the existence of a nonempty core.[10]

2 Environment

Let $N = \{1, \dots, n\}$ be a finite set of agents, considered fixed in what follows. A network is an undirected graph that is a list of unordered pairs of agents $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked under the network g. When it is unambiguous, I write ij to represent $\{i, j\}$. Let g^S be the complete network (the set of all subsets of S of size two) on $S \subseteq N$. Denote by $G = \left\{g \mid g \subset g^N\right\}$ the set of all possible networks on N. Furthermore, let $N(g)$ be the set of agents who have at least one link in g.

A *path* in a network $g \in G$ between agents i and j is a sequence of agents $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ for some $K \geq 2$, with $i_1 = i$ and $i_K =$

[10]Note that this paper does not intend to provide any *normative* statements for such relations. For example, the West cooperated with Hitler, Mussolini, and Franco when its enemy of the 1930s was Stalin (Saperstein (2004)). Therefore, such a condition for stability does not justify any normative arguments for peace. Rather, this paper provides one game-theoretic explanation for the reason why the dichotomy of political alliances such as the Cold War can persist for a long time.

j. A *component* of a network g is a nonempty subnetwork $g' \subset g$ such that (1) if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and (2) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$. Denote by $C(g)$ the set of components of g.

A *value function* is a function $v : G \to \mathbb{R}$ that determines the total value $v(g)$ for each network $g \in G$. The set of all possible value functions is denoted by V. As noted by Jackson (2005), the value function contains a characteristic function of a cooperative game as a special case, since it allows the value that accrues to depend on both the coalition of agents involved and the network structure. A value function v is *component additive* if $v(g) = \sum_{g' \in C(g)} v(g')$ for any $g \in G$. This paper restricts attention to the "interesting sub-class of value functions" in which the value of a given component of a network is independent of the structure of other components (Jackson (2005), p. 132). This precludes externalities across (but not within) components of a network.

A network $g \in G$ is *efficient* relative to a value function v if $v(g) \geqslant v(g')$ for all $g' \in G$. Given a value function v, its *monotonic cover* $\hat{v}$ is defined by $\hat{v}(g) = \max_{g' \subset g} v(g')$. A monotonic cover is by definition *monotonic* in that $\hat{v}(g) \leq \hat{v}(g')$ for $g \subset g'$. A *network game* is a pair (N, v).

An *allocation rule* is a function $Y : G \times V \to \mathbb{R}^n$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g. An allocation rule determines how the value generated by a network is allocated among the agents, either through their decisions or even by some outside intervention. A network allocation pair $g \subset g^N$ and (imputation) $y \in \mathbb{R}^n$ is in the *core* of the network game (N, v) if $\sum_i y_i \leqslant v(g)$ (feasibility) and $\sum_{i \in S} y_i \geqslant \hat{v}(g^S)$ (coalitional relationality), where g^S is the complete network on S, for all $S \subset N$. An allocation rule Y is *core consistent* if, for any v such that the core is nonempty, there exists at least one g such that $(g, Y(g, v))$ is in the core.

To describe sufficient conditions and obtain the existence result, I impose more structure on value functions. The exogenously given surplus value of a link among every pair of agents from the perspective of agent i is represented by a symmetric *weight matrix* or interchangeably *valuation graph* of i, W^i, where $w^i_{ij} \in \mathbb{R}$ for $i \neq j$ captures the value of a link with agent j from the perspective of i, while $w^i_{jk} \in \mathbb{R}$ for $i \neq j \neq k$ represents a pairwise complementarity between potential peers j and k for i. I call an element of a valuation graph an *edge weight*. An edge weight w^i_{ij} indicates an *actual* value of forming a link with j for i. I assume that $w^i_{ii} = w^i_{jj} = 0$ for any i and j and $w^i_{jk} = w^i_{kj}$.

A *signed graph* for agent i is comprised of a pair $R^i = (N, W^i)$. The *underlying graph* of R^i is (N, E^i), where E^i contains a *value edge* (i, j) whenever $w^i_{ij} \neq 0$ and a value edge (j, k) whenever $w^i_{jk} \neq 0$, which is not same as a link in a network but rather is essentially an exogenously given link in a valuation graph. Notice that the value of each pair of agents can differ among different agents captured by differences between W^i and W^j for $i \neq j$, to account for heterogeneity in valuation and the costs of maintaining such relations.

I assume BQP preference structures on value function such that

$$\sum_{i \in N} \left(\sum_{j:\{i,j\} \in g} w^i_{ij} + \sum_{j,k:\{i,j\},\{i,k\} \in g} w^i_{jk} \right) = v(g),$$

where w^i_{jk} does not need to be zero even when j and k do not form a relation.

3 Existence

I first describe the so-called sign-consistency assumption introduced by Candogan et al. (2015).

Assumption 3.1. *(Sign Consistency). For some* $i, j \in N$*, if* $w^i_{ij} > 0$*, then* $w^k_{ij} \geq 0$ *for all* $k \in N$*, and similarly, if* $w^i_{ij} < 0$*, then* $w^l_{ij} < 0$ *for all* $l \in N$*.*

For example, if the relation between Japan and South Korea is negative, I assume that it has nonpositive synergistic effects on all other countries when these countries actually form a relation with both Japan and South Korea. While it is entirely possible that a country may welcome the tension between the two countries, I assume that if this country actually forms a relation with both of the countries, then this country inevitably incurs the negative synergy.

Next, I introduce the so-called sign-balance assumption.[11]

Assumption 3.2. *(Sign Balance). For all* $i \in N$*, each cycle in* $\boldsymbol{R}^i$ *has an even number of negative edge weights.*

Figure 1 shows examples of a sign-balanced graph. The two graphs on the left are sign balanced, while the two on the right are not. Colloquially, the condition requires that the enemy of my enemy is my friend.

My proof for the main result exploits the primal-dual relation between welfare-maximizing solutions and the core. In particular, I first show that the following quadratic program (QP1) has an integer-value solution[12]

[11] This is sometimes called the "structural balance condition (see, e.g., Altafini (2012))."

[12] Following the standard of the engineering literature, I put optimization programming indexing such as (QP1) in front of the optimization problem to distinguish it from equation numbering.

$$(\mathrm{QP1})\, H(N) = \quad \text{maximize} \sum_{k\in N}\left(\sum_{i\neq k} w_{ik}^k x_i^k + \sum_{i\neq j\neq k} w_{ij}^k x_i^k x_j^k\right)$$

$$\text{subject to } x_i^k = x_k^i \ \ \forall i,k \in N,$$

$$0 \le x_i^k \le 1 \ \ \forall i,k \in N,$$

where $x_i^k = 1$ if agent i forms a relation with agent k, $0 < x_i^k < 1$ if a *fraction* of agent i and k form a relation, and $x_i^k = 0$ if agent i does not form a relation with agent k. Note that if a fraction of agent i and k form a link, its interpretation is purely mathematical and has no real-life meaning; for example, if $x_i^k = 0.4$, 40% of i and k form a link. For convenience, define $x_k^k = 0$ for all $k \in N$. Note that from an integral solution to (QP1), we can construct a network.

The important property of a sign-balanced graph is so-called *clusterability* (Cartwright and Harary (1956)). Fix $i \in N$. The clusterability means that nodes of R^i can be grouped into two *disjoint* sets S_1 and S_2, within which edge weights between any two nodes are nonnegative,

Notice that by sign consistency, the property implies that all agents can be grouped into L_1 and L_2 under R^j for any $j \neq i$ meaning that all agents can be grouped into L_1 and L_2. Let $E^+ = \{(i,j) : w_{ij}^k \ge 0 \text{ for any } k \in N\}$ and $E^- = \{(i,j) : w_{ij}^k < 0 \text{ for any } k \in N\}$. If either $L_1 = \emptyset$ or $L_2 = \emptyset$, then the complete network is a solution which is integral. If both are the empty sets, then a zero vector is the solution and therefore, the solution is integral.

Thus, suppose L_1 and L_2 are nonempty sets. The goal is to show the integrality of a solution to (QP1) and construct a dual that corresponds to the core of the game. To show the integrality, I first introduce a new variable, z_{ij}^k for each triplet i, j, k, to linearly relax the quadratic terms in (QP1)[13]

$$
\begin{aligned}
\text{(LP1)} \quad \text{maximize} \sum_{k \in N} \left(\sum_{i \neq k} w_{ik}^k x_i^k + \sum_{i \neq j \neq k} w_{ij}^k z_{ij}^k \right) \\
\text{subject to } x_i^k \leq 1 \ \ \forall i, k \in N \\
x_i^k = x_k^i \ \ \forall i, k \in N \\
z_{ij}^k \leq x_i^k, x_j^k \ \ \forall k \in N, (i, j) \in E_+^k \\
z_{ij}^k \geq x_i^k + x_j^k - 1 \ \ \forall k \in N, (i, j) \in E_-^k \\
x_i^k, z_{ij}^k \geq 0 \ \ \forall i, j, k \in N
\end{aligned}
$$

We call this linearly relaxed formulation (LP1). Let x be a vector of all x_i^k with $|N| \times |N|$ dimension and z be a vector of all z_{ij}^k with $|N| \times |N| \times |N|$ dimension. Let $o = |N|^2 + |N|^3$. We present the constraints of (LP1) can be written as $At \leq b$, where t is a column vector of $[x \ z]$, A is a matrix with o-by-o dimension whose elements a_{qp} are the coefficients of choice variables in (LP1) which take value of 0 or 1, and b is a column vector of constants that take 0 or 1 with o dimension in (LP1).

A polyhedron $P \subseteq \mathbb{R}^o$ is the set of all points $t \in \mathbb{R}^o$ that satisfy a finite set of linear inequalities, which can be mathematically expressed as

$$P = \left\{ t \in \mathbb{R}^o \, : \, At \leq b \right\}.$$

[13]See Bertsimas et al. (1999) for a similar relaxation method in quadratic Boolean optimization problems at Section 2.2 of their paper.

Note that both the feasible set and solution set of a linear program are certain polyhedra. A matrix is called *totally unimodular* if and only if the determinant of each square submatrix has value 1, -1 or 0. A is a *network matrix* if its entry $a_{ij} = 0, 1,$ or -1 for all i, j and each column contains at most two non-zero entries of opposite signs. I will use the celebrated Hoffman-Kruskal theorem from Hoffman and Kruskal (1956) that a solution to a linear program is integral when its constraint matrix is totally unimodular, and a matrix is totally unimodular when it is a network matrix as implied by Tutte (1965).

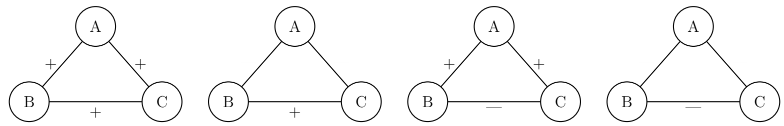
I prove the existence of an integral solution to (LP1) by extending the version of the proof for the tree-valuation graph of Candogan et al. (2015) in one of Vohra's blog posts (2014).[14] The proof for the following lemma can be found in the Appendix. Note that the following lemma is not implied by Bertsimas et al. (1999). They deal with surplus maximization of one agent with a finite number of goods, while I study total welfare maximization of finitely many agents in a network game.

Lemma 1. *Let Assumptions 3.1 and 3.2 hold. Then, (LP1) has an integral solution.*

Take an extreme point solution to (LP1), $(\bar{z}, \bar{x})$. Then, since (LP1) is a linearly relaxed version of (QP1), integral solutions coincide between the two programs, and thus $\bar{x}$ is an extreme point solution to (QP1) as well. Notice that since $\bar{x}$ is a binary vector that takes 0 or 1 for each pair of agents, we can determine which pair forms a link based on $\bar{x}$. That is, we start with g^N, keep $\{i, j\}$ pair if $x^i_j = 1$ and take out from g^N if $x^i_j = 0$ for any $i \neq j \in N$. After doing this procedure for every pair of agents, we are left with g such that $v(g) = \hat{v}(g^N) = H(N)$.

[14] https://theoryclass.wordpress.com/2014/02/10/combinatorial-auctions-and-binary-quadratic-valuations-postscript/

Figure 1: Examples of a cycle in balanced and unbalanced graphs



The two graphs on the left are balanced, and the ones on the right are unbalanced.

The integrality of a solution to (LP1) and thus (QP1) actually implies the nonemptiness of the core. Deng et al. (1999) demonstrate that the core of a coalitional game is nonempty if and only if the linear programming formulation of the game has an integral solution. The following lemma is implied by Theorem 1 of their paper.[15]

Theorem 1. *Let Assumption 3.1 and 3.2 hold. Then, the core is nonempty.*

One may wonder if the core of a coalitional game in this paper is nonempty even without the sign-balance condition. The example at the Appendix demonstrates that this is not the case. Therefore, the sign-balance condition is indeed significant.

4 Conclusion

Focusing on network games that involve pairwise complementarities between peers, this paper provides a novel sufficient condition for the nonemptiness of the core of a network games that involve pairwise complementarities between peers that allows a new class of preferences relevant to political alliances. The paper's limitation is the lack of characterization of the core.

[15]For readers to whom the application of Theorem 1 in Deng et al. (1999) is not immediate, my own proof is available upon request.

5 Declarations

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References

Altafini, C. (2012). “Consensus problems on networks with antagonistic interactions”. *IEEE transactions on automatic control*, 58(4):935–946.

Ausubel, L. M., Cramton, P., McAfee, R. P., and McMillan, J. (1997). “Synergies in wireless telephony: Evidence from the broadband pcs auctions”. *Journal of Economics & Management Strategy*, 6(3):497–527.

Bertsimas, D., Teo, C., and Vohra, R. (1999). “On dependent randomized rounding algorithms”. *Operations Research Letters*, 24(3):105–114.

Candogan, O., Ozdaglar, A., and Parrilo, P. (2018). “Pricing equilibria and graphical valuations”. *ACM Transactions on Economics and Computation (TEAC)*, 6(1):2.

Candogan, O., Ozdaglar, A., and Parrilo, P. A. (2015). “Iterative auction design for tree valuations”. *Operations Research*, 63(4):751–771.

Cartwright, D. and Harary, F. (1956). “Structural balance: a generalization of heider’s theory.” *Psychological review*, 63(5):277.

Deng, X., Ibaraki, T., and Nagamochi, H. (1999). “Algorithmic aspects of the core of combinatorial optimization games”. *Mathematics of Operations Research*, 24(3):751–766.

Facchetti, G., Iacono, G., and Altafini, C. (2011). “Computing global structural balance in large-scale signed social networks”. *Proceedings of the National Academy of Sciences*, 108(52):20953–20958.

Hoffman, A. and Kruskal, J. (1956). “Integral boundary points of convex polyhedra, in “linear inequalities and related systems”(hw kuhn and aw tucker, eds.)”. *Annals of Mathematical Studies*, 38.

Jackson, M. O. (2005). “Allocation rules for network games”. *Games and economic behavior*, 51(1):128–154.

Jackson, M. O. and Wolinsky, A. (1996). “A strategic model of social and economic networks”. *Journal of economic theory*, 71(1):44–74.

Maoz, Z., Terris, L. G., Kuperman, R. D., and Talmud, I. (2007). “What is the enemy of my enemy? causes and consequences of imbalanced international relations, 1816–2001”. *The Journal of Politics*, 69(1):100–115.

Nakamura, K., Tita, G., and Krackhardt, D. (2019). “Violence in the “balance”: a structural analysis of how rivals, allies, and third-parties shape inter-gang violence”. *Global Crime*, pages 1–25.

Navarro, N. (2007). “Fair allocation in networks with externalities”. *Games and Economic Behavior*, 58(2):354–364.

— (2010). “Flexible network rules for identified externalities”. *Games and Economic Behavior*, 69(2):401–410.

Saperstein, A. M. (2004). ““the enemy of my enemy is my friend” is the enemy: Dealing with the war-provoking rules of intent”. *Conflict Management and Peace Science*, 21(4):287–296.

Shapley, L. and Scarf, H. (1974). “On cores and indivisibility”. *Journal of mathematical economics*, 1(1):23–37.

Shapley, L. S. (1955). “Markets as cooperative games”. Technical report, RAND CORP SANTA MONICA CA.

Szell, M., Lambiotte, R., and Thurner, S. (2010). “Multirelational organization of large-scale social networks in an online world”. *Proceedings of the National Academy of Sciences*, 107(31):13636–13641.

Tomita, H. (1995). “A sign of liberalization in the 1960’s egypt”. *Journal of law, politics, and sociology.*

Tutte, W. T. (1965). “Lectures on matroids”. *Journal of Research of the National Bureau of Standards Section B Mathematics and Mathematical Physics*, page 1.

Appendix

Proof of Lemma 1

Proof. Before delving into the proof, let us define a network matrix. Matrix $\boldsymbol{A}$ is a network matrix if its entry $a_{ij} = 0, 1$, or -1 for all i, j and each column contains at most two non-zero entries of opposite sign. Notice that since the decision variables of (LP1) are all between 0 and 1, the feasible set is bounded. Since the feasible set is bounded, there exists an extreme point solution to (LP1). Denote it by $(\bar{z}, \bar{x})$. The goal is to show that $(\bar{z}, \bar{x})$ is indeed integral.

As explained in the main text, by sign consistency and sign balancedness, all agents can be grouped into two disjoint sets L_1 and L_2, which which edge weights between any two nodes are nonnegative and across which edge weights between any two nodes are strictl negative. If either $L_1 = \emptyset$ or $L_2 = \emptyset$, then the complete network is a solution to our network game. This implies that the following program (LPL1), which is (LP1) when the set of agents is restricted to L_1, has an extreme point solution that is a vector of 1s:

$$
\begin{aligned}
\text{(LP1L1)} \quad \text{maximize } \bar{v}_1(x) \equiv \sum_{k \in L_1} \left(\sum_{i \neq k} w_{ik}^k x_i^k + \sum_{i \neq j \neq k} w_{ij}^k z_{ij}^k \right) \\
\text{subject to } x_i^k \leq 1 \ \ \forall i, k \in L_1 \\
x_i^k = x_k^i \ \ \forall i, k \in L_1 \\
z_{ij}^k \leq x_i^k, x_j^k \ \ \forall k \in L_1, (i, j) \in E_+^k \\
x_i^k, z_{ij}^k \geq 0 \ \ \forall i, j, k \in L_1,
\end{aligned}
$$

where x is a $|L_1|^3$ dimension vector whose entries are x_i^j for $i, j \in L_1$. We can define (LPL2) in a similar manner by relabelling L_1 with L_2, which has a vector of 1s as a solution. Denote by $barv_2$ the objective function of (LPL2). If both are the empty sets, then a zero vector is the solution and therefore, the solution is integral. Therefore, let us focus on the case when both sets are nonempty.

Let P be the solution set of (LPL1) and let P' be that of (LPL2). Now, let $\{X_1, ..., X_n\}$ be the set of extreme points of P for some natural number n while $\{Y_1, ..., Y_{n'}\}$ be that of P' for some natural number n'.In fact, if all edge weights within the members of L_1 and L_2 are non-zero, then there is only one extreme point solution for each of (LPL1) and (LPL2). Since a polyhedron is convex, all values of fractional solutions to (LPL1) can be expressed by a convex combination of $\bar{v}_1$, and so are (LPL2) by a convex combination of $\bar{v}_2$. Let $\sum_{r=1}^{n} \lambda_r \bar{v}(X_r)$ be that for (LPL1) and $\sum_{r'=1}^{n'} \zeta_r \bar{v}(Y_{r'})$ be that for (LPL2), where $\sum_1^n \lambda_r = 1$ and $\sum_1^{n'} \zeta_{r'} = 1$.

Since the objective function of (LP1) is linear in its arguments, and since a polyhedron is convex, the objective functional values of all fractional solutions and integral solutions to (LP1) can be expressed as the sum of these convex combinations plus the sum of negative weights between (p, q) for all $p \in L_1$ and $q \in L_2$ pairs with decision variables $0 \leq y_{pq}^k \leq 1$. Let E_- be the set of negative edges restricted to those involving the vertices in L_1. Then, we can reformulate (LP1) into the following linear program:

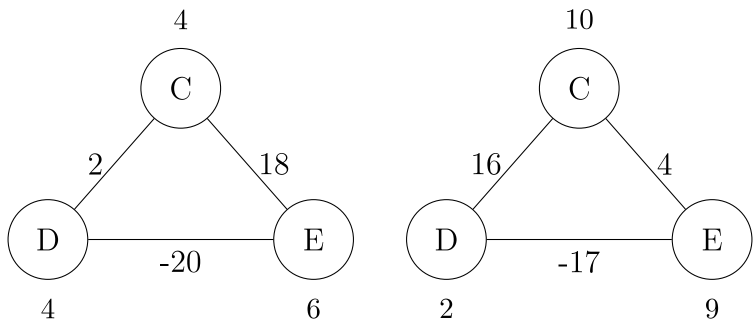
$$
\begin{aligned}
\text{maximize} & \sum_{r=1}^{n} \lambda_r \bar{v}_1(X_r) + \sum_{r'=1}^{n'} \zeta_r \bar{v}_2(Y_{r'}) - \sum_{k \in L_1} \sum_{(p,q) \in E_-} |w_{pq}^k| y_{pq}^k \\
\text{subject to} & - \sum_{1}^{n} \lambda_r = -1 \\
& - \sum_{1}^{n'} \zeta_r = -1 \\
& y_{pq}^k \leq 1 \quad \forall k \in L_1, (p,q) \in E_- \\
& \lambda_r, \zeta_r, y_{pq}^k \geq 0 \ \forall r, k,
\end{aligned}
$$

Notice that the constraint matrix of this linear program is a network matrix. This is because each variable appears in at most one constraint with a coefficient of 1 or -1. A network matrix is totally unimodular as implied by Tutte (1965). Since a linear program with a totally unimodular constraint matrix has an integral solution as shown by Hoffman and Kruskal (1956), there exists an integral solution in this program,. This implies that a combination of some extreme points X_r and Y_r together with $\{y_{pq}^k\}$ which is integral maximizes the objective value of the above program. Notice that this combination can be expressed by $\bar{t}$, implying that $\bar{t}$ is an integral solution as well. ■

Example when Sign-Balancedness Fails

In this section, I shall show how violation of the sign-balance condition can result in an empty core. Suppose we have two agents A and B with $w_{AB}^A = w_{AB}^B = -100$. Furthermore, suppose there are thee more agents, C, D, and E with which agent A or B is considering forming a relation, while agents D, E, and F are indifferent in forming relations with each other—i.e.,

Figure 2: Example without a core



The left graph corresponds to agent A's valuation graph when restricted to agent C, D, and E, and the right graph corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$.

$w^k_{\text{CD}} = w^k_{\text{CE}} = w^k_{\text{DE}} = 0$ for $k \in \{\text{C, D, E}\}$. Moreover, from these three agents' perspectives, there is no intrinsic benefit of forming a relation with A or B—i.e., $w^k_{\text{A}k} = w^k_{\text{B}k} = 0$ for $k \in \{\text{D, E, F}\}$. Finally, assume that w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$ and w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$ are depicted by Figure 2. The graph on the left corresponds to agents A's value graph when restricted to agent C, D, and E, and the graph on the right corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$. In this example, any of the three agents C, D, and E will not form a coalition with *both* A and B, due to the prohibitively high negative synergy between A and B.

This example has an empty core. Just to illustrate how the infinite loop of blocking occurs, let us look at an arbitrary start of this loop. Suppose agent A forms a relation with C and E together, while agent B forms a relation with agent D. At a glance, this seems to be an efficient outcome and thus achieves no blocking coalition. And yet, notice that if agent A pays agent C less than 26, then agent C forms a blocking coalition with agent B and D.

So, agent A has to pay agent C 26, and pays agent E no more than 2 since otherwise, agent A would obtain a negative payoff. But then, agent B will leave agent D and form a blocking coalition with agent E, paying her any amount in (2, 7) (since agent B can get at most 2 from matching with agent D).

Similar blocking processes will happen at any combination of coalition formation among these five agents, and thus this is an example with an empty core when the sign-balance condition is violated.